Journal of Engineering Physics and Thermophysics, Vol. 71, No. 5, 1998

USE OF THE METHOD OF GREEN FUNCTIONS FOR NUMERICAL SOLUTION OF MULTIDIMENSIONAL STEFAN PROBLEMS

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UDC 536.25:517.95

It is shown that use of an auxiliary Green function, namely, the Green function of the boundary-value problem for the Laplace operator with a condition of the third kind artificially introduced on part of the boundary, makes it possible to find numerical solutions of multidimensional Stefan problems with any boundary conditions. The efficiency of the method is verified for a Stefan problem that has an exact solution.

In [1] a numerical-analytical method is suggested that makes it possible to find approximate solutions of one-dimensional nonstationary Stefan problems with allowance for the dependence of the coefficients of the heat conduction equation on the temperature. The essence of this approach consists in successive use of Kirchhoff's transform of the sought temperature, the method of straight lines, and the method of Green functions. A numerical solution of the obtained system of Hammerstein-type integral equations is sought by the projection-grid (zonal) method with retention of the stable computational grid in each time step. As compared to difference schemes with explicit separation of the front [2-4] the method of [1] is distinguished by logical simplicity, and in economy it compares favorably with the difference algorithm of through calculation [5, 6]. In what follows, we consider the special features of application of the approach of [1] to the solution of multidimensional nonstationary Stefan problems.

The differences in the formulation and the algorithm of the solution of the Stefan problem in the one- and multidimensional cases are already fully revealed in a comparison of the one- and two-dimensional cases. Therefore, we restrict our considerations to the Stefan problem with two spatial variables. For simplicity, we restrict ourselves to the case of the two-phase Stefan problem, although the method is suitable for any number of phases. We consider only boundary conditions that require construction of an auxiliary Green function [1], thus covering the cases of both nonlinear and linear boundary conditions of the second and third kind. Stefan problems with boundary conditions of the first kind (when the law of temperature variation is assigned on a portion of the boundary) are solved by means of an ordinary Green function.

Assuming that straight lines parallel to the Ox axis intersect the phase interface at just one point, i.e., its equation $\Phi(x, y, t) = 0$ can be represented in the form $x = z(y, t)$, we write a general formulation of the nonstationary Stefan problem with two spatial variables for the cases of a rectangular region ($k = 0$) and axial symmetry of the temperature field $T(x, y, t)$ ($k = 1$, the region is a finite cylinder):

$$x^{-k} \frac{\partial}{\partial x} \left(x^k \lambda(T) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(\lambda(T) \frac{\partial T}{\partial y} \right) = \gamma(T) \frac{\partial T}{\partial t} - w(x, y, t, T) +$$

$$+ p x^k |z_t(y, t)| \, \delta_{(k)}(x - z(y, t)) , \quad 0 < x < a , \quad 0 < y < b , \quad t > 0 ; \quad k = 0 ; 1 ; \tag{1}$$

$$T(x, y, 0) = T^0(x, y) ; \quad T(z(y, t), y, t) = T_* ; \quad z(y, 0) = z_0(y) , \quad 0 < z_0(y) < a ;$$

$$\lambda(T) T_x = - \varepsilon_k q_0(y, t, T) , \quad x = 0 ; \quad \lambda(T) T_x = q_1(y, t, T) , \quad x = a ;$$

I. Franko Lvov State University, Lvov, Ukraine. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 71, No. 5, pp. 910-916, September-October, 1998. Original article submitted February 12, 1997.

$$\lambda(T)\,T_y = -p_0(x,t,T),\ \ y=0;\ \ \lambda(T)\,T_y = p_1(x,t,T),\ \ y=b.$$

Here $\varepsilon_0 = 1$; $\varepsilon_1 = 0$; $p = \text{const}$, $p > 0$ in the case of melting and $p < 0$ in crystallization, $T_* = \text{const}$ is the temperature of the phase transition. The assigned functions $\lambda(T)$, $\gamma(T)$, and $w(x, y, t, T)$ can have a jumpwise discontinuity in T in passing through the phase interface, liable to determination along with the temperature field.

Assuming $T(x, y, t) \geq T_0 = \text{const}$ and having used the Kirchhoff transform

$$u(x,y,t) = u(T) = \int_{T_0}^{T(x,y,t)} \lambda(T)\,dT$$

we obtain the Stefan problem in a form convenient for subsequent solution to determine the function $z(y, t)$ and the modified temperature $u(x, y, t)$:

$$\Delta_k u = \Gamma(u)\,u_t - W(x,y,t,u) + p\,|z_t(y,t)|\,x^k\,\delta_{(k)}(x - z(y,t)),\ \ 0<x<a,$$

$$0<y<b,\ \ t>0;$$

$$u(x,y,0) = u_0(x,y);\ \ u(z(y,t),y,t) = u_*;\ \ z(y,0) = z_0(y); \tag{2}$$

$$u_x = -\varepsilon_k Q_0(y,t,u),\ \ x=0;\ \ u_x = Q_1(y,t,u),\ \ x=a;$$

$$u_y = -P_0(x,t,u),\ \ y=0;\ \ u_y = P_1(y,t,u),\ \ y=b.$$

Here $\Gamma(u) = \gamma(T(u))dT(u)/du$; $W(x, y, t, u) = w(x, y, t, T(u))$; $Q_i(y, t, u) = q_i(y, t, T(u))$; $P_i(x, t, u) = p_i(x, t, T(u))$; $i = 0, 1$; $u_* = u(T_*)$, $T(u)$ is the inverse function to $u(T)$.

To solve problem (2) approximately we use the method of straight lines [7, 8]. Having introduced the notation $t_m = m\tau$, $u_m(x, y) = u(x, y, t_m)$, $z_m(y) = z(y, t_m)$, $W_m(x, y, u_m) = W(x, y, t_m, u_m)$, $Q_{im}(y, u_m) = Q_i(y, t_m, u_m)$, $P_{im}(x, u_m) = P_i(x, t_m u_m)$, $i = 0, 1$, we obtain a nonlinear boundary-value problem for determining the function $u_m(x, y)$ on the current time layer $t = t_m$:

$$\Delta_k u_m = \Gamma(u_m)(u_m - u_{m-1})/(\sigma\tau) - W_m(x,y,u_m) + px^k\,|z_m(y) - z_{m-1}(y)|\,\delta_{(k)}(x -$$

$$- z_m(y))/(\sigma\tau) - (1-\sigma)(\Delta_k u_{m-1} + W_{m-1}(x,y,u_{m-1}))/\sigma,\ \ 0<x<a;\ \ 0<y<b; \tag{3}$$

$$u_{mk} = -\varepsilon_k Q_{0m}(y,u_m),\ \ x=0;\ \ u_{mx} = Q_{1m}(y,u_m),\ \ x=\dot{a};$$

$$u_{my} = -P_{0m}(x,u_m),\ \ y=0;\ \ u_{my} = P_{1m}(x,u_m),\ \ y=b;\ \ m=1,2,\ldots,$$

and an additional condition for determining $z_m(y)$:

$$u_m(z_m(y), y) = u_*;\ \ m = 1,2,\ldots. \tag{4}$$

We show that the boundary-value problem (3), (4) approximates the Stefan problem (2) at the time $t = t_m$ with an error $O(\tau)$ at $\sigma = 1$ and an error $O(\tau^2)$ at $\sigma = 1/2$. For $x \neq z_m(y)$ the validity of this assertion is obvious [8]. Having introduced the notation $\Phi_m(x, y) = x - z_m(y)$ we consider a point P on the surface $\Phi_m = 0$ and construct a cylinder of rather small volume with center at the point P and generatrix parallel to the normal $n(P)$ to $\Phi_m = 0$ at the point P. This cylinder is symmetric relative to the surface $\Phi_m = 0$. Let S_1 be the side surface and S_2^- and S_2^+ be, respectively, the lower and upper base of the cylinder; their areas are denoted, respectively, by $|S_1|$ and $|S_2| = |S_2^-| = |S_2^+|$. We multiply the differential equation of problem (3) by x^k, integrate it over the cylinder volume, and let $|S_1|$ and then $|S_2|$ tend to zero. After transformations of the volume integrals similar to ones performed in [5, pp. 818-819], we obtain in the limit the relation

$$((\mathrm{grad}\, u_m)_{P+0} - (\mathrm{grad}\, u_m)_{P-0},\ \mathrm{grad}\, \Phi_m) = p\ |\Phi_m(x, y) - \Phi_{m-1}(x, y)|/(\sigma\tau) -$$

$$- (1 - \sigma)\left\{((\mathrm{grad}\, u_{m-1})_{P+0} - (\mathrm{grad}\, u_{m-1})_{P-0},\ \mathrm{grad}\, \Phi_m)\right\}/\sigma,$$

which is an approximation for $t = t_m$ of the well-known Stefan condition

$$((\mathrm{grad}\, u)_{P+0} - (\mathrm{grad}\, u)_{P-0},\ \mathrm{grad}\, \Phi) = p\ |\Phi_t|,$$

with an error $O(\tau)$ at $\sigma = 1$ and an error $O(\tau^2)$ at $\sigma = 1/2$. Here the subscripts $P + 0$ and $P - 0$ denote the limiting value of the quantities in approaching the point P lying on the phase front from the regions with lower and higher heat contents [6].

Since the Green function of the linear boundary-value problem corresponding to (3) (the second boundary-value problem for the Laplace operator) does not exist [9], then, having represented one of the boundary conditions of problem (3), e.g., at $x = a$, in the form

$$u_{mx} + hu_m = Q_{1m}(y, u_m) + hu_m, \quad x = a,$$

(h is an arbitrary positive number), we consider an auxiliary Green function $G_k(x, y; \xi, \eta)$ as the solution of the following boundary-value problem:

$$\Delta_k G_k(x, y; \xi, \eta) = -\delta_{(k)}(x - \xi)\,\delta(y - \eta), \quad 0 < x < a, \quad 0 < y < b;$$
$$G_{kx} = 0, \quad x = 0; \quad G_{ky} = 0, \quad y = 0; \quad G_{kx} + hG_k = 0, \quad x = a; \quad G_{ky} = 0, \quad y = b. \tag{5}$$

We find from (5)

$$G_k(x, y; \xi, \eta) = \sum_{n=1}^{\infty} \left\{G_n(y, \eta)\,\varphi_{kn}(\xi)\,\varphi_{kn}(x)/\Phi_{kn}\right\}, \quad k = 0;\ 1; \tag{6}$$

where for $k = 0$ (the plane problem)

$$\varphi_{0n}(x) = \cos(\gamma_n x), \quad \Phi_{0n} = (\gamma_0^2 a + h\cos^2(\gamma_n a))\,(1 - \exp(-2\gamma_n b))/\gamma_n,$$

$$\gamma_n > 0, \quad \mathrm{cotan}\,(\gamma_n a) = \gamma_n/h, \quad n = 1, 2, \ldots,$$

and for $k = 1$ (the axisymmetric problem)

$$\varphi_{1n}(x) = J_0(\gamma_n x), \quad \Phi_{1n} = a^2(\gamma_n^2 + h^2)\,(1 - \exp(-2\gamma_n b))\,J_0^2(\gamma_n a)/\gamma_n,$$

$$\gamma_n > 0, \quad hJ_0(\gamma_n a) = \gamma_n J_1(\gamma_n a), \quad n = 1, 2, \ldots.$$

In what follows, to simplify the presentation we write Δ, G, φ_n, Φ_n instead of Δ_k, G_k, φ_{kn}, F_{kn}.

Having used the second Green formula for the operator Δ applied to the functions u_m and G, we obtain an integral representation for the solution $u_m(x, y)$ of the boundaty-value problem (3):

$$u_m(x, y) = \int_0^a [G(x, y; \xi, b)\,P_{1m}(\xi, u_m(\xi, b)) + G(x, y; \xi, 0)\,P_{0m}(\xi, u_m(\xi, 0))] \times$$

$$\times \xi^k d\xi + \int_0^b \left\{a^k G(x, y; a, \eta)\,[hu_m(a, \eta) + Q_{1m}(\eta, u_m(a, \eta))] + \varepsilon_k G(x, y; 0, \eta) \times\right.$$

$$\times Q_{0m}(\eta, u_m(0,\eta))\Big\} d\eta - \int_0^a \int_0^b G(x, y; \xi, \eta) F_m(\xi, \eta, u_m(\xi, \eta)) \xi^k d\xi d\eta -$$

$$- p \int_{y_{0m}}^{y_{1m}} G(x, y; z_m(\eta), \eta) \, |z_m(\eta) - z_{m-1}(\eta)| \, z_m^k(\eta) \, d\eta/(\sigma\tau), \quad 0 \le x \le a, \quad 0 \le x \le b, \tag{7}$$

where $F_m(x, y, u) = \frac{\Gamma(u)(u - u_{m-1}}{\sigma\tau} - W_m(x, y, u) - \frac{(1-\sigma)(\Delta u_{m-1} + W_{m-1}(x, y, u_{m-1}))}{\sigma}$;

y_{0m}, y_{1m} are the ordinates of the points of intersection of the curve $x = z_m(y)$ for $x \in [0, a]$ with Oy axis; $y_{1m} > y_{0m}$; we suppose that there can be no more than two such points. Here the following cases are possible: $y_{0m} = 0$, $y_{1m} = b$ (there are no intersection points at all), $y_{0m} = 0$, $y_{1m} \in (0, b]$ and $y_{0m} \in [0, b)$, $y_{1m} = b$ (only one intersection point exists). The nonlinear integral equation (7) should be solved simultaneously with condition (4).

The solution of Eq. (7) is sought in the form of a series in the eigenfunctions $\varphi_n(x)$ of expansion (6) of the Green function $G(x, y; \xi, \eta)$:

$$u_m(x, y) = \sum_{n=1}^{\infty} g_{nm}(y) \varphi_n(x). \tag{8}$$

To solve the problem approximately, in series (6) and (8) we restrict ourselves to the sums of the first N terms (N is rather large), and then after substitution of these partial sums in Eq. (7) we obtain a system of N nonlinear integral equations:

$$g_{nm}(y) = \Bigg\{ \int_0^a [G_n(y, b) P_{1m}(\xi, v_{1m}(\xi)) + G_n(y, 0) P_{0m}(\xi, v_{0m}(\xi))] \varphi_n(\xi) \xi^k d\xi +$$

$$+ \int_0^b \Big\{ a^k \varphi_n(a) [h w_{1m}(\eta) + Q_{1m}(\eta, w_{1m}(\eta))] + \varepsilon_k \varphi_n(0) Q_{0m}(\eta, w_{0m}(\eta)) \Big\} G_n(y, \eta) \, d\eta -$$

$$- \int_0^a \int_0^b G_n(y, \eta) \varphi_n(\xi) F_m(\xi, \eta, \sum_{s=1}^{N} g_{sm}(\eta) \varphi_s(\xi)) \xi^k d\xi d\eta -$$

$$- p \int_{y_{0m}}^{y_{1m}} G_n(y, \eta) \varphi_n(z_m(\eta)) \, |z_m(\eta) - z_{m-1}(\eta)| \, z_m^k(\eta) \, d\eta/(\sigma\tau) \Bigg\} \Big/ \Phi_n; \quad 0 \le y \le b; \quad n = \overline{1, N}, \tag{9}$$

where

$$v_{0m}(x) = \sum_{s=1}^{N} g_{sm}(0) \varphi_s(x), \quad v_{1m}(x) = \sum_{s=1}^{N} g_{sm}(b) \varphi_s(x),$$

$$w_{0m}(y) = \sum_{s=1}^{N} g_{sm}(y) \varphi_s(0), \quad w_{1m}(y) = \sum_{s=1}^{N} g_{sm}(y) \varphi_s(a).$$

Equations (9) are a system of integral equations of the Hammerstein type in $g_{nm}(y)$, $n = \overline{1, N}$, which, moreover, involves the unknown constants $g_{nm}(0)$ (for $k = 0$), $g_{nm}(b)$, $n = \overline{1, N}$, y_{0m}, y_{1m} and the unknown function $z_m(y)$. First of all this system should be complemented by $(\varepsilon_k + 1)N$ equations, having set $y = 0$ and $y = b$ successively in (9). Then we obtain a system of $(\varepsilon_k + 2)N$ equations in $g_{nm}(y)$, $g_{nm}(0)$, and $g_{nm}(b)$, $n = \overline{1, N}$. As concerns the unknowns y_{0m}, y_{1m}, and $z_m(y)$, in a first approximation they can be taken equal to the known values obtained in the preceding time layer, and then, as necessary, we can refine the solution by an iteration technique.

It is convenient to seek a numerical solution of the obtained system of $(\varepsilon_k + 2)N$ nonlinear integral equations by a projection-grid zonal method [10], which makes it possible to reduce it to a system of $(\varepsilon_k + 1 + M)N$ nonlinear algebraic equations in $g_{nm}(0)$ (for $k = 0$), $g_{nm}(b)$, and the values of the functions $g_{nm}(y)$

$$g_{nmi} = (k+1) \int_{y_{i-1}}^{y_i} g_{nm}(y)\, y^k dy / (y_i^{k+1} - y_{i-1}^{k+1}), \quad i = \overline{1, M}, \quad n = \overline{1, N},$$

averaged over the intervals (y_{i-1}, y_i) of subdivision of the segment $[0, b]$. Having solved this system by one of the effective iteration techniques, e.g., the Newton method, using the solution of the system in the preceding time layer as the initial approximation, and having approximately calculated in (9) the integrals of the form

$$\int_0^b f(\eta, g_{nm}(\eta))\, d\eta \approx \sum_{i=1}^{M} \int_{y_{i-1}}^{y_i} f(\eta, g_{nmi})\, d\eta,$$

we can use relations (9) to determine the functions $g_{nm}(y)$. Then we find the coordinates of the points of the phase interface by means of condition (4), viz., we determine y_{0m}, y_{1m} as the solution of the equation

$$\sum_{n=1}^{N} g_{nm}(y)\, \varphi_n(0) = u_*$$

for $y \in [0, b]$. Then, dividing the segment $[y_{0m}, y_{1m}]$ into K parts, we seek $x_{mi} = z_m(y_{mi})$, where $y_{mi} = y_{0m} + i(y_{1m} - y_{0m})/K$, $i = \overline{0, K}$, from the equations

$$\sum_{n=1}^{N} g_{nm}(y_{mi})\, \varphi_n(x) = u_*, \quad i = \overline{0, K-1}.$$

each of which, according to the assumption made above, has a single root for $x \in [0, a]$. The function $z_m(y)$ can then be interpolated using the values $x_{mi} = z_m(y_{mi})$ found.

In the case where the functions q_0, q_1, p_0, p_1 in the boundary conditions of problem (1) are independent of T (linear boundary conditions), and the dependences of the functions $\lambda(T)$, $\gamma(T)$, and $w(x, y, t, T)$ on T have the piecewise-constant form

$$\beta(T) = \begin{cases} \beta_s = \text{const}, & T < T_*, \\ \beta_L = \text{const}, & T \geq T_*, \end{cases} \quad \beta = \lambda; \gamma; \quad w(x, y, t, T) = \begin{cases} w_s(x, y, t), & T < T_*, \\ w_L(x, y, t), & T \geq T_*, \end{cases} \tag{10}$$

the system of integral equations (9) becomes linear in $g_{nm}(y)$ and does not involve $g_{nm}(0)$ or $g_{nm}(b)$.

The suggested technique of numerical solution of two-dimensional nonstationary Stefan problems is checked by comparing results of calculations with the exact solution

$$T(x, y, t) = B_L - A_L (x^2 + y^2)/(t_0 - t), \quad 0 \leq x \leq z(y, t); \quad 0 \leq y \leq y(t);$$

$$T(x, y, t) = B_s - A_s (x^2 + y^2)/(t_0 - t), \quad \begin{cases} z(y, t) \leq x \leq a, & 0 \leq y \leq y(t), \\ 0 \leq x \leq a, & y(t) \leq y \leq b; \end{cases}$$

$$z(y, t) = (\alpha^2 (t_0 - t) - y^2)^{1/2}, \quad y(t) = \alpha (t_0 - t)^{1/2}, \quad 0 \leq t < t_0;$$

$$A_s = (4\lambda_L (B_L - T_*) + p\alpha^2)/(4\lambda_s \alpha^2); \quad A_L = (B_L - T_*)\alpha^{-2}; \quad B_s = T_* + \alpha^2 A_s$$

TABLE 1. Comparison of Results of Calculation of Values of $y(t)$ with the Exact Solution

t	$y(t)$				
	$N=60$, $M=4, \tau=1$	$N=60$, $M=6, \tau=1$	$N=80$, $M=4, \tau=1.5$	$N=80$, $M=4, \tau=1$	exact solution
3	0.9399	0.9397	0.9343	0.9377	0.9381
6	0.8791	0.8791	0.8648	0.8730	0.8717
9	0.8105	0.8101	0.7850	0.8001	0.8000
12	0.7277	0.7272	0.6882	0.7135	0.7211
15	0.6214	0.6217	0.5606	0.6247	0.6325

TABLE 2. Comparison of Results of Calculation of Values of the Function $z(y, t)$ Obtained at $\tau = 1$, $\delta = 0.5$, $N = 80$, $t = 10$ with the Exact Solution

y	$z(y, t)$	
	calculated value	exact value
0.0000	0.7796	0.7746
0.0773	0.7758	0.7707
0.1546	0.7643	0.7590
0.2319	0.7443	0.7391
0.3092	0.7154	0.7102
0.3865	0.6761	0.6713
0.4638	0.6253	0.6204
0.5411	0.5596	0.5542
0.6184	0.4710	0.4664
0.6957	0.3457	0.3405
0.7730	0.0000	0.0497
0.7746	–	0.0000

of the problem with a phase interface of circular shape [5] that we write in the form of (1) with $k = 0$, $q_0(y, t, T) = p_0(x, t, T) = 0$, $q_1(y, t, T) = -2A_s\lambda_s a/(t_0 - t)$, $p_1(x, t, T) = -2A_s\lambda_s b/(t_0 - t)$, $w(x, y, t, T) = A(T)(4\lambda(T)(t_0 - t) - \gamma(T)(x^2 + y^2))(t_0 - t)^{-2}$, where $\lambda(T)$, $\gamma(T)$, and $A(T)$ have the form of (10).

The calculations were performed for: $t_0 = 25$; $T_* = 0$; $B_L = 1$; $\alpha = 0.2$; $p = -1$; $\lambda_s = 0.5$; $\lambda_L = 0.75$; $\gamma_s = 2$; $\gamma_L = 1.25$; $a = b = 2$; $\sigma = 1$; $h = 1$; a time step $\tau = 1$ and $\tau = 1.5$; a step in the spatial variable $\delta = 0.5$ ($M = 4$) and $\delta = 0.33$ ($M = 6$). In solving the system for g_{nmi}, $i = \overline{1, M}$, $n = \overline{1, N}$, we used the values of $z_m(y)$ from the preceding time layer. The accuracy of the results of calculation depends on the value of the auxiliary parameter h, since h affects the rate of series convergence of the series and to obtain the same accuracy for different h we must take different values of N. The results of calculations presented in Tables 1-3 show that this method makes it possible to obtain a rather accurate solution. As in the case of one spatial variable [1], a reduction in the time step gives better results than the same reduction in the spatial-variable step.

Thus, the suggested numerical-analytical method for solving multidimensional nonstationary Stefan problems is distinguished by logical simplicity, does not reqire reconstruction of the computational grid in each

TABLE 3. Comparison of Results of Calculation of Values of $T(x, y, t)$ Obtained at $\tau = 1$, $\delta = 0.5$, $N = 80$, $t = 5$, 10, 15 with the Exact Solution (upper row – calculated value; lower row – exact value)

y	x		
	0	1	2
		$t = 5$	
0	1.0098	−0.3747	−6.0978
	1.0000	−0.3800	−6.0800
1	−0.3566	−2.2557	−7.9641
	−0.3800	−2.2800	−7.9800
2	−5.9139	−7.7824	−13.1663
	−6.0800	−7.9800	−13.6800
		$t = 10$	
0	1.0110	−1.0146	−8.6625
	1.0000	−1.0133	−8.6133
1	−0.9893	−3.5298	−11.1584
	−1.0133	−3.5467	−11.1467
2	−8.4317	−10.9315	−18.1271
	−8.6133	−11.1467	−18.7467
		$t = 15$	
0	0.9429	−2.4178	−13.9535
	1.0000	−2.2800	−13.6800
1	−2.3823	−6.2276	−17.7343
	−2.2800	−6.0800	−17.4800
2	−13.6256	−17.4109	−28.2671
	−13.6800	−17.4800	−28.8800

time step, and is suited for any boundary conditions and for regions admitting construction of the Green function of the first or third boundary-value problem for the Laplace operator.

NOTATION

T, temperature; x, y, spatial coordinates; t, time; T_*, temperature of the phase transition; T^0, initial temperature distribution; λ, coefficient of thermal conductivity; $\gamma = c\rho$; c, specific heat capacity; ρ, density; w, distribution of internal heat sources; $x = z(y, t)$, equation for the phase interface; $y(t)$, ordinate of the point of intersection of the curve $x = z(y, t)$ with the Oy axis; $p = L\rho_L(T_*)$; L, latent heat of the phase transition; ρ_L, density of the liquid phase; q_0, q_1, p_0, p_1, heat fluxes at the boundaries $x = 0$, $x = a$, $y = 0$, $y = b$, respectively; $\delta(x)$, Dirac delta function; $\delta_{(k)}(x)$, Dirac delta function with weight x^k; $\Delta_k u = x^{-k}(x^k u_x)_x + u_{yy}$, Laplace operator for the cases of plane ($k = 0$) and cylindrical ($k = 1$) symmetry; $u_x = \partial u/\partial x$; $u_{yy} = \partial^2 u/\partial y^2$; $J_0(x)$, $J_1(x)$, Bessel functions of the first kind and zero and first orders, respectively.

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